Answer Sheet 1

1. $x + 15 = (x - 15)^2$ $x + 15 = x^2 - 30x + 225$ $x^2 - 31x + 210 = 0$

   Using the quadratic equation formula,

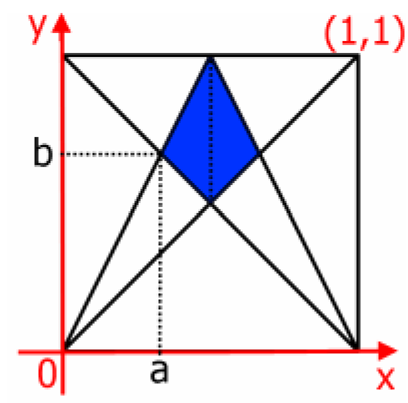
   $x = \frac{31 \pm \sqrt{31^2 - 4 \times 1 \times 210}}{2 \times 1}$

   $x = \frac{(31+11)}{2} = 21$ or $x = \frac{(31-11)}{2} = 10$

   As he had a birthday 15 years ago, he cannot be 10 years old so his age is 21.

2. Coordinates method (can also be solved with pythagoras or similar triangles)

   Consider a unit square drawn on a coordinate grid.



   The line joining $(0, 0)$ to $(1, 1)$ has equation $y = x$. The line joining $(0, 0)$ to $(\frac{1}{2}, 1)$ has equation $y = 2x$ The line joining $(0, 1)$ to $(1, 0)$ has equation $y = 1 - x$.

   The point (a, b) is at the intersection of the lines $y = 2x$ and $y = 1 - x$.

   So $a = \frac{1}{3}, b = \frac{2}{3}$.

   The shaded area is made up of two congruent triangles, one of which has vertices $(\frac{1}{3}, \frac{2}{3}), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 1)$.

   The perpendicular height of the triangle is $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

   Area of the triangle $= \frac{1}{2}(\frac{1}{2} \times \frac{1}{6}) = \frac{1}{24}$

   Therefore the shaded area is $2 \times \frac{1}{24} = \frac{1}{12}$

3. We take the square root symbol in the question to signify the positive square root. The tactic here is to square both sides and then find the correct square root. If $\sqrt{2 + \sqrt{3}} - \sqrt{2 - \sqrt{3}} = X$, then

$$X^2 = \left(\sqrt{2 + \sqrt{3}} - \sqrt{2 - \sqrt{3}}\right)\left(\sqrt{2 + \sqrt{3}} - \sqrt{2 - \sqrt{3}}\right).$$

   The right hand side equals

$$2 + \sqrt{3} - 2\left(\sqrt{2 + \sqrt{3}} \times \sqrt{2 - \sqrt{3}}\right) + 2 - \sqrt{3},$$

and

$$\sqrt{2+\sqrt{3}} \times \sqrt{2-\sqrt{3}} = 1.$$

Therefore

$$X^2 = 2+\sqrt{3} - 2 + 2 - \sqrt{3} = 2.$$

Does $X = -\sqrt{2}$ or $+\sqrt{2}$?

Well $2+\sqrt{3} > 2-\sqrt{3}$, so $\sqrt{2+\sqrt{3}} > \sqrt{2-\sqrt{3}}$, so X is positive and we have $X = \sqrt{2}$.

In the second part there are again many solutions (because square roots have two values and cube roots have three values). To simplify the solution we restrict ourselves to real cube roots. We want to find

$$X = \sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}.$$

One way to do this is to write $a = \sqrt[3]{2+\sqrt{5}}$ and $b = \sqrt[3]{2-\sqrt{5}}$, and use the equation

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + 3ab(a+b) + b^3.$$

As $X = a + b$, we have

$$X^3 = (2+\sqrt{5}) + 3X\sqrt[3]{(2+\sqrt{5})(2-\sqrt{5})} + (2-\sqrt{5}).$$

As $\sqrt[3]{(2+\sqrt{5})(2-\sqrt{5}} = \sqrt[3]{-1} = -1$ this gives $X^3 + 3X - 4 = 0$ and hence

$$(X-1)(X^2 + X + 4) = 0.$$

As $X^2 + X + 4 = 0$ has only complex solutions, and we are looking for the real values of X, so we have $X = 1$, that is

$$\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}} = 1.$$

4. *S*tudents should be able to form the equation but might not have a method to solve it. The solution below uses Euclid's Algorithm, so you may wish to go through it on the board with them.

   Let x be the three digit number at the start. Let y be the four digit number at the end of the phone number. The original phone number is $10000x + y$. The changed phone number is $1000y + x$. The new number is one more than the old number doubled so

$$20000x + 2y + 1 = 1000y + x$$

$$19999x + 1 = 998y.$$

$$\begin{aligned}
& & (1)\\
\mathbf{19,999} &= 998 \times 20 + 39 & (2)\\
\mathbf{998} &= \mathbf{39} \times 25 + 23 & (3)\\
\mathbf{39} &= \mathbf{23} \times 1 + 16 & (4)\\
\mathbf{23} &= \mathbf{16} \times 1 + 7 & (5)\\
\mathbf{16} &= \mathbf{7} \times 2 + 2 & (6)\\
\mathbf{7} &= \mathbf{2} \times 3 + 1 & (7)\\
\mathbf{2} &= \mathbf{1} \times 2 + 0. & (8)
\end{aligned}$$

Working backwards to get values for x and y:

$$\begin{aligned}
1 &= \mathbf{7} - 3 \times \mathbf{2} & (9)\\
&= \mathbf{7} - 3 \times (\mathbf{16} - 2 \times \mathbf{7}) & (10)\\
&= 7 \times \mathbf{7} - 3 \times \mathbf{16} & (11)\\
&= 7 \times (\mathbf{23} - \mathbf{16}) - 3 \times \mathbf{16} & (12)\\
&= 7 \times \mathbf{23} - 10 \times \mathbf{16} & (13)\\
&= 7 \times \mathbf{23} - 10 \times (\mathbf{39} - \mathbf{23}) & (14)\\
&= 17 \times \mathbf{23} - 10 \times \mathbf{39} & (15)\\
&= 17 \times (\mathbf{998} - 25 \times \mathbf{39}) - 10 \times \mathbf{39} & (16)\\
&= 17 \times \mathbf{998} - 435 \times \mathbf{39} & (17)\\
&= 17 \times \mathbf{998} - 435 \times (\mathbf{19999} - 20 \times \mathbf{998}) & (18)\\
&= 8717 \times \mathbf{998} - 435 \times \mathbf{19999}. & (19)
\end{aligned}$$

Thus $y = 8717$ and $x = 435$. The old telephone number is therefore 4358717.

5. Suppose $x^2 = 100a + 10b + c$ where a, b and c are whole numbers, $a \geq 1$ and b and c are between 0 and 9 inclusive.

$$x^2 - 9 = (x-3)(x+3) = \star\star\star\star 9000$$

As 10 divides the right hand side of this expression we know 10 divides $x-3$ or $x+3$. Thus x ends in a 3 or a 7.

All that remains is to try the different cases with x=3 and x=7, working mod 100, to find the solution 1503.

6. $81 + 36 + 4 = 121; 9 + 6 + 2 = 17$

7. There are a variety of different solutions that can be verified quite easily!